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MINIMAL ENTROPY RIGIDITY FOR FOLIATIONS OF COMPACT SPACES

BY

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ABSTRACT

We formulate and prove a foliated version of a theorem of Besson, Courtois, and Gallot establishing the minimal entropy rigidity of negatively curved locally symmetric spaces. One corollary is a foliated version of Mostow's rigidity theorem.

1. Introduction

One version of the minimal entropy rigidity theorem of Besson, Courtois, and Gallot says that a compact negatively curved locally symmetric space uniquely minimizes normalized volume growth entropy among all negatively curved manifolds homotopy equivalent to it. More precisely, let (X, g_o) be a compact negatively curved locally symmetric space of dimension $n \geq 3$, and let g be any negatively curved metric on a compact space Y homotopy equivalent to X. For any $y \in \tilde{Y}$, we define the quantities,

$$\bar{h}(g) = \limsup_{R\to\infty} \frac{1}{R} \log(\operatorname{Vol} B(y,R)) \quad \text{and} \quad \underline{h}(g) = \liminf_{R\to\infty} \frac{1}{R} \log(\operatorname{Vol} B(y,R)),$$

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where $B(y, R)$ is the geodesic ball of radius R about y in $\tilde{Y}$. The quantities $\bar{h}(g)$ and $\underline{h}(g)$ are independent of the choice of $y \in \tilde{Y}$. Manning [Man79] showed that $\bar{h}(g) = \underline{h}(g)$ and this quantity is called the **volume growth entropy** $\mathrm{h}(g)$. Minimal entropy rigidity then states

THEOREM 1 (Besson–Courtois–Gallot [BVG96]): *With the above notations,*

$$h(g_o)^n \operatorname{Vol}(X, dg_o) \leq h(g)^n \operatorname{Vol}(Y, dg),$$

and equality is achieved if and only if g is homothetic to g_o.

In this paper we prove a foliated version of Theorem 1. We let N and M be compact topological manifolds supporting continuous foliations $\mathcal{F}_N$ and $\mathcal{F}_M$ by leaves which are smooth Riemannian manifolds, and such that the metrics on the leaves vary continuously in the transverse direction. The role of the locally symmetric space is played by $(M, \mathcal{F}_M)$, for which we suppose that the leaves are locally isometric to n-dimensional symmetric spaces of negative curvature, $n \geq 3$. (By continuity of the metrics these are all locally homothetic to a fixed symmetric space $(\tilde{X}, g_o)$.)

For the foliation $(N, \mathcal{F}_N)$ we assume that the leaves (L, g_L) are strictly negatively curved, and satisfy a stronger condition (that they are Patterson–Sullivan manifolds) which we define below. Finally, the role of the homotopy equivalence is played by a leaf-preserving homeomorphism

$$f\colon (N, \mathcal{F}_N) \to (M, \mathcal{F}_M)$$

which is leafwise C^1 with transversally continuous leafwise derivatives. (Note that we do not assume that f is transversally differentiable.)

Our first step is to introduce a class of manifolds which we call **Patterson–Sullivan manifolds**. Consider a negatively curved manifold (L, g_L) and equip its universal cover $\tilde{L}$ with a **uniform tiling** by domains of bounded diameter and volume (see §2 for the precise definition). Compact manifolds are prime examples to keep in mind, where the tiling is by Dirichlet fundamental domains. A (not necessarily compact) leaf of a continuous foliation of a compact space also provides a natural example, where the tiling is by the lifts to $\tilde{L}$ of the foliation plaques (see §2 for definitions about foliations).

Using the tiling, on $\tilde{L}$ we construct Patterson–Sullivan measures ν_x, one for each $x \in \tilde{L}$. Even if $\bar{h}(g_L) \neq \underline{h}(g_L)$, there is a distinguished number $h(g_L)$ satisfying $\underline{h}(g_L) \leq h(g_L) \leq \bar{h}(g_L)$ which we call the **volume growth entropy**. Fixing a base point $p \in \tilde{L}$, we say that L is a **Patterson–Sullivan manifold** if, for $x \in \tilde{L}$, the total mass at infinity $\nu_x(\partial\tilde{L})$ of the Patterson–Sullivan measures as a function of $d(p,x)$ has exponential growth/decay less than $\mathrm{h}(g_L)$ (see §2 for the precise definition).

If $\tilde{L}$ is cocompact, then $\nu_x(\partial\tilde{L})$ is equivariant with respect to the action of the fundamental group, so is actually bounded away from zero and infinity. However, when L is a general noncompact space with an arbitrary uniform tiling, the irregularity of the tiles prevents a priori bounds on $\nu_x(\partial\tilde{L})$.

Now let L be a (not necessarily compact) Patterson–Sullivan manifold and $f\colon L \to X$ a homeomorphism from L to a negatively curved manifold. Suppose some lift $\tilde{f}\colon \tilde{L} \to \tilde{X}$ is a quasi-isometry. (In §4 we show that this is true when L is a leaf of a compact foliation and f is the restriction to L of the foliation homeomorphism.) As in [BCG96], we take the barycenter of the pushforward Patterson–Sullivan measures and construct a natural map $\tilde{F}\colon \tilde{L} \to \tilde{X}$, which descends to the quotients. Our first result, which is is the key to our foliated version of Theorem 1, is

THEOREM 2: *The map $\tilde{F}$ is a proper surjection.*

When $\tilde{L}$ has a compact quotient (e.g., in Theorem 1), then Theorem 2 is a trivial consequence of degree theory, since $\tilde{F}$ descends to a map F on the compact quotients which is homotopic to the original homotopy equivalence $f\colon L \to X$. In the foliation case we are interested in, we apply Theorem 2 to each leaf and prove a global foliated coarea formula which allows us to prove the Main Theorem.

For any metric space (L, g_L), we may define the quantities $\bar{h}(g_L)$ and $\underline{h}(g_L)$ as before. We define the volume growth entropy $h(g_L)$ as

$$h(g_L) = \inf\left\{ s > 0 \,\middle|\, \int_0^\infty e^{-st} \operatorname{Vol} S(x,t)\,dt < \infty \right\},$$

where $S(x,t)$ is the sphere of radius t about x in the universal cover $\tilde{L}$ of L. This quantity is independent of $x \in L$ and so, when L is a leaf of the foliation on N, $L \mapsto h(g_L)$ is a function from N to $[0,\infty]$ which is constant on each leaf. In fact, by volume comparison with constant negatively curved spaces we observe that $h(g_L)$ must lie in the range $[(n-1)a, (n-1)b]$, when the sectional curvatures of L are bounded between $[-b^2, -a^2]$.

The function $h(g_L)$ is also measurable on N because the transverse continuity of the leafwise metrics implies that for each R, the function

$$x \mapsto \int_0^R e^{-st} \operatorname{Vol} S(x,t)\,dt$$

is continuous on N. On $(M, \mathcal{F}_M)$ the entropy is constant and we denote it by $h(g_o)$. Since the exponential volume growth of balls is governed by the exponential growth of spheres, we may replace $B(y, R)$ with $S(y, R)$ in the definition of $\bar{h}(g_L)$ and $\underline{h}(g_L)$ without change. Then from the definition of $h(g_L)$ it is clear that $\underline{h}(g_L) \leq h(g_L) \leq \bar{h}(g_L)$.

Equip the foliation $(N, \mathcal{F}_N)$ with any choice of finite transverse holonomy quasi-invariant measure ν (see Hurder [Hur94] or Zimmer [Zim82] for the definition and existence). Holonomy quasi-invariance simply means that the push forward of ν under any holonomy map is in the same measure class as ν. This measure ν provides us with a global finite measure μ_N on N which is locally a product of ν with the Riemannian volumes dg_L of the leaves L.

Main Theorem: *Let $(N, \mathcal{F}_N)$ be a continuous foliation of the compact manifold N such that ν-almost every leaf is a Patterson–Sullivan manifold. Suppose that $f\colon (N, \mathcal{F}_N) \to (M, \mathcal{F}_M)$ is a foliation-preserving homeomorphism, leafwise C^1 with transversally continuous leafwise derivatives, and that $f_*\nu$-almost every leaf of $(M, \mathcal{F}_M)$ is a rank one locally symmetric space. Then there exists a finite measure μ_M on M which is locally the product of dg_o with a transverse quasi-invariant measure ν_o such that*

$$\int_M h(g_o)^n d\mu_M \leq \int_N h(g_L)^n d\mu_N,$$

and equality holds if and only if ν-almost every leaf (L, g_L) is homothetic to its image $(f(L), g_o)$.

When the foliation admits a holonomy invariant measure ν, then we may take $\nu_o = f_*\nu$. When ν is just holonomy quasi-invariant however, then ν_o is the push forward of ν under the natural map F defined below.

When the foliation $(N, \mathcal{F}_N)$ is ergodic with respect to ν, then the entropy function $h(g_L) = h(g)$ is constant on N, and we get the

Corollary 1.1: *Under the same assumptions as in the main theorem, if $(N, \mathcal{F}_N)$ is ergodic, then $h(g_o)^n \operatorname{Vol}(M, \mu_M) \leq h(g)^n \operatorname{Vol}(N, \mu_N)$ with equality if and only if ν-almost every leaf (L, g_L) is homothetic to $(f(L), g_o)$.*

Remarks.:

1. If $(N, \mathcal{F}_N)$ and $(M, \mathcal{F}_M)$ are foliations such that almost every leaf is compact or simply connected, then the requirement that the homeomorphism f be leafwise C^1 can be dropped. In particular, if the foliations have just one leaf and $\dim N \neq 3, 4$, any homotopy equivalence induces a homeomorphism

between N and M (see [FJ93]). Therefore, when $\dim N \neq 3, 4$, Corollary 1.1 recovers Theorem 1.

2. In fact, Theorem 1 is true in greater generality, namely when the metric g on Y is *any* (not necessarily negatively curved) metric and when X and Y are related by a map of non-zero degree (see [BCG96]). We conjecture that a foliated version of this more general theorem is also true, and would yield interesting results about foliations.
3. In Section 2.1 we give examples of classes of foliations $(N, \mathcal{F}_N)$ where almost every leaf is a Patterson–Sullivan manifold. It is also an open question whether this assumption is unnecessary. It seems that the "recurring geometry" imposed on leaves of compact foliations by the recurrence of the leaves inside the ambient space might already give strong bounds on the mass at infinity of Patterson–Sullivan measures.

The outline for the paper is as follows. In §2 we construct Patterson–Sullivan measures on Patterson–Sullivan manifolds. In §3 we describe the construction of the natural map on such manifolds, and prove Theorem 2. In §4 we show that, in our foliation case, any lift $\tilde{f}$ to the universal covers of the leaves is a quasi-isometry, so that we are in a position to apply Theorem 2 to each leaf. In §5 we prove a foliated coarea formula which, together with a crucial estimate from [BCG96] on the Jacobian of the natural map, allows us to derive the main theorem. Lastly, in §6 we present some applications of the Main Theorem, including a foliated version of Mostow's rigidity theorem.

2. Patterson–Sullivan measures and manifolds

Let (L, g_L) be a negatively curved manifold.

Definition 2.1: A countable partition $\{D_j\}_{j=1}^{\infty}$ of the universal cover $\tilde{L}$ is a **uniform tiling** if there exist constants $C_1 > 1$ and $C_2 > 0$ such that for all j,

1. $C_1^{-1} < \mathrm{Vol}(D_j) < C_1$, and
2. $\mathrm{Diam}\, D_j < C_2$.

Choosing one point d_j from each D_j, we call the collection $\{d_j\}$ a **lattice** Λ associated to the tiling.

We point out that uniform tilings always exist for negatively curved manifolds since we do not require the tiles to have bounded inscribed radius.

Let us now be specific about how a leaf in a compact foliation has a naturally defined uniform tiling. For this we recall some definitions for foliations (from

[Hur94]). An n-dimensional continuous foliation $\mathcal{F}$ of codimension q on the paracompact manifold N^{n+q} is a partition of N^{n+q} into a set of C^∞ manifolds, the **leaves**, of dimension n with some additional structure. Namely, if $D^i(r)$ denotes the open ball of radius r in $\mathbb{R}^i$, we require that

1. there exist a uniformly locally-finite open cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of N^{n+q},
2. there exist homeomorphisms $\phi_\alpha\colon U_\alpha \to D^q(1) \times D^n(1)$ which extend to homeomorphisms $\tilde{\phi}_\alpha\colon \tilde{U}_\alpha \to D^q(2) \times D^n(2)$ where $\tilde{U}_\alpha$ contains the closure of U_α, and
3. for each $x \in D^q(2)$, the set $\tilde{\phi}_\alpha^{-1}(\{x\} \times D^n(2))$ is the connected component containing $\tilde{\phi}_\alpha^{-1}(\{x\} \times \{0\})$ of the intersection of $\tilde{U}_\alpha$ with the leaf through $\tilde{\phi}_\alpha^{-1}(\{x\} \times \{0\})$.

Such a set of charts $\{U_\alpha, \phi_\alpha\}_{\alpha\in\mathcal{A}}$ is a **regular foliation atlas** for $\mathcal{F}$. The topological disks $\phi_\alpha^{-1}(\{x\} \times D^n(1))$ are **plaques**, and the U_α are **flow boxes**.

Restricting our attention to the manifold N, compactness allows us to choose an atlas consisting of finitely many flow boxes $\{U_i\}_{i=1}^m$ for $\mathcal{F}_N$. A **transversal** is a Borel subset $T \subset N$ which intersects each leaf of the foliation in at most a countable set. Given an atlas it is natural to choose transversals which intersect each plaque exactly once. We can always do this by taking **local cross sections** $T_i = \phi_i^{-1}(D^q \times \{x\})$ for any $x \in D^n$ from which we obtain a **complete transversal** $T = \bigcup_i T_i$.

Let (L, g_L) be a leaf of $(N, \mathcal{F}_N)$, $\tilde{L}$ its universal cover, and $\pi\colon \tilde{L} \to L$ the covering map. Since the metrics on the leaves vary continuously in the foliation, the leafwise plaque diameters are globally bounded from above and below away from zero, and similarly their volumes as well. These plaques form a locally finite open cover of L, so we may choose a partition of L subordinate to this cover whose lift forms a uniform tiling of the universal cover $\tilde{L}$. For the lattice we take the natural choice, $\Lambda = \pi^{-1}(T \cap L)$, the lifts to $\tilde{L}$ of the points in the leaf L where the transversal T meets L.

Returning to the more general discussion of manifolds with a given uniform tiling and associated lattice Λ, we now construct the Patterson–Sullivan measures on them. Fix a basepoint $p \in \tilde{L}$ and let $d(x,y)$ be the distance function on $\tilde{L}$. Consider the Poincaré series

$$g_s(x) = \sum_{y\in\Lambda} q(d(x,y))e^{-sd(x,y)},$$

where $q(t)\colon \mathbb{R}^+ \to \mathbb{R}^+$ is any nondecreasing function (to be determined shortly) such that for any $\epsilon > 0$ and $d > 0$ there is an $r > 0$ such that

$$\left|\frac{q(r+d)}{q(r)} - 1\right| < \epsilon.$$

The corresponding truncated series for $R > 0$ will be denoted by,

$$g_s(x, R) = \sum_{y \in \Lambda \cap B(x,R)} q(d(x,y))e^{-sd(x,y)}.$$

By the assumptions on the diameter and volume of each tile, for all $R > 0$ and all $s > 0$,

$$\begin{aligned} C_1^{-1}e^{-sC_2} \int_{B(x,R)} q(d(x,y))e^{-sd(x,y)} dg_{\tilde{L}}(y) &\leq g_s(x,R) \\ \leq C_1 e^{sC_2} \int_{B(x,R)} q(d(x,y))e^{-sd(x,y)} dg_{\tilde{L}}(y). \end{aligned} \tag{2.1}$$

Since the integrals and $g_s(x, R)$ are non-decreasing in R, we can take limits to obtain

$$\begin{aligned} C_1^{-1}e^{-sC_2} \int_0^\infty q(t)e^{-st} \operatorname{Vol}(S(x,t))dt \\ \leq g_s(x) \leq C_1 e^{sC_2} \int_0^\infty q(t)e^{-st} \operatorname{Vol}(S(x,t))dt. \end{aligned} \tag{2.2}$$

By the definition of $\mathrm{h}(g_L)$, (2.2) shows that $g_s(x)$ diverges for $s < \mathrm{h}(g_L)$ and converges for $s > \mathrm{h}(g_L)$. Patterson ([Patterson76]) showed that for any Poincaré series there is a choice of a weighting function q such that $g_s(x)$ diverges at $s = \mathrm{h}(g_L)$; we make this choice of q. Hence the above integrals also diverge at $s = \mathrm{h}(g_L)$.

For $s > \mathrm{h}(g_L)$ we form the measures

$$\nu_x^s = \frac{\sum_{y \in \Lambda} q(d(x,y))e^{-sd(x,y)}\delta_y}{g_s(p)},$$

on $\tilde{L}$ where δ_y is the Dirac delta measure at y. For one $x \in \tilde{L}$ we may take a weak limit of these measures along a fixed sequence $s_i \to \mathrm{h}(g_L)^+$ to obtain the measure

$$\nu_x = \lim_{s_i \to \mathrm{h}(g_L)^+} \nu_x^s.$$

Since the series $g_{\mathrm{h}(g_L)}(x)$ diverges it follows that the measure ν_x is supported on a subset of the boundary $\partial\tilde{L}$. As noted by Sullivan [Sul79], for any other

$y \in \tilde{L}$, the same weak limit also converges to a measure ν_y, which is absolutely continuous with respect to ν_x. To see this we compute the Radon–Nikodym derivative explicitly.

For any $\xi \in \partial\tilde{L}$ which is in the support of ν_x, let $\{B_\epsilon\}_{0<\epsilon<1}$ be a family of open sets in $\tilde{L} \cup \partial\tilde{L}$ such that $B_\epsilon \subset B_{\epsilon'}$ whenever $\epsilon < \epsilon'$, $\cap_{\epsilon>0} B_\epsilon = \{\xi\}$ and $B_\epsilon \cap \partial\tilde{L}$ is open in $\partial\tilde{L}$ for all ϵ. Hence $\nu_x(B_\epsilon) > 0$ for all $x \in \tilde{L}$ and $\epsilon > 0$. It follows from the Radon–Nikodym theorem that

$$\begin{aligned}\frac{d\nu_x}{d\nu_y}(\xi) &= \lim_{\epsilon\to 0} \lim_{s_i \to \mathrm{h}(g_L)^+} \frac{\nu_x^{s_i}(B_\epsilon)}{\nu_y^{s_i}(B_\epsilon)} \\ &= \lim_{\epsilon\to 0} \lim_{s_i \to \mathrm{h}(g_L)^+} \frac{\sum_{z\in\Lambda\cap B_\epsilon} q(d(x,z))e^{-s_i d(x,z)}}{\sum_{z\in\Lambda\cap B_\epsilon} q(d(y,z))e^{-s_i d(y,z)}}.\end{aligned}$$

Since B_ϵ contracts to ξ and by definition of the Busemann function $B(x,y,\xi)$, for each $z \in \Lambda \cap B_\epsilon$ we have $d(y,z) = d(x,z) + B(x,y,\xi) + \delta(z,\epsilon)$ where $|\delta(z,\epsilon)| \to 0$ as $\epsilon \to 0$ for all $z \in \Lambda \cap B_\epsilon$. Plugging this into the previous equation above and using the properties of the weighting function q, we obtain

$$\begin{aligned}\frac{d\nu_x}{d\nu_y}(\xi) &= \lim_{s_i \to \mathrm{h}(g_L)^+} e^{s_i B(x,y,\xi)} \\ &= e^{\mathrm{h}(g_L)B(x,y,\xi)}.\end{aligned} \tag{2.3}$$

By construction, the measures ν_x are equivariant under any covering isometries γ of the leaf: $\gamma_* \nu_x = \nu_{\gamma x}$.

Letting $c(x) = \nu_x(\partial\tilde{L})$ be the total mass of ν_x, we define the **normalized Patterson–Sullivan measures** to be the probability measures

$$\mu_x = \frac{\nu_x}{c(x)}.$$

They are equivariant under isometries and satisfy

$$\frac{d\mu_x}{d\mu_y}(\xi) = \frac{c(y)}{c(x)} e^{\mathrm{h}(g_L)B(x,y,\xi)}. \tag{2.4}$$

Definition 2.2: Let (L, g_L) be a negatively curved manifold, Λ a lattice associated to a uniform tiling of $\tilde{L}$, and ν_x the associated Patterson–Sullivan measures. L is a Patterson-Sullivan manifold if

$$\limsup_{x\in\tilde{L}} \left| \frac{\log c(x)}{d(p,x)} \right| < h(g_L).$$

It is easy to check that this definition is independent of the choice of basepoint p. We point out that a simple estimate using the triangle inequality on the Poincaré series shows that

$$\limsup_{x\in\tilde{L}}\left|\frac{\log c(x)}{d(p,x)}\right|\leq h(g_L)$$

always holds. In the case when L is compact, $c(x)$ descends to a smooth (and hence bounded) function on L, so the left hand side is zero.

As we will see later, a sufficient condition for this Patterson–Sullivan condition in more geometric terms is that for all $x\in\tilde{L}$,

$$\limsup_{d(x,y)\to\infty}\limsup_{R\to\infty}\frac{\log(\operatorname{Vol}S(x,R)/\operatorname{Vol}S(y,R))}{d(x,y)}<h(g_L),$$

where the outer lim sup runs over all sequences of $y\in\tilde{L}$ tending to the boundary.

We will need later (in constructing the natural map) that the measures μ_x do not have atoms. Clearly it is enough to check this for ν_x. For this, take a sequence x_n along a geodesic ray with endpoints x and θ, and (2.3) shows that

$$\begin{aligned}\nu_{x_n}(\partial\tilde{L})=\int_{\partial\tilde{L}}e^{-\,\mathrm{h}(g_L)B(x,x_n,\xi)}d\nu_x(\xi)&\geq e^{-\,\mathrm{h}(g_L)B(x,x_n,\theta)}\nu_x(\theta)\\&=e^{\mathrm{h}(g_L)d(x,x_n)}\nu_x(\theta).\end{aligned}$$

If θ is an atom of ν_x, then $\nu_{x_n}(\partial\tilde{L})$ has exponential growth rate $h(g_L)$, contradicting the assumption that L is a Patterson–Sullivan manifold.

Similarly, for Patterson–Sullivan manifolds we can show that ν_x (and hence μ_x) is supported on all of $\partial\tilde{L}$. For if not, then take a sequence x_n converging to a point in the complement of the support. Since the complement is open, for any $\xi\in\partial\tilde{L}$ in the support of ν_x, there exists a constant $C>0$ and $N>0$ such that for $n>N$, $e^{-h(g_L)B(x,x_n,\xi)}\leq Ce^{-h(g_L)d(x,x_n)}$. Hence,

$$\nu_{x_n}(\partial\tilde{L})=\int_{\partial\tilde{L}}e^{-\,\mathrm{h}(g_L)B(x,x_n,\xi)}d\nu_x(\xi)\leq Ce^{-\,\mathrm{h}(g_L)d(x,x_n)}\nu_x(\partial\tilde{L}).$$

However, this exceeds the allowable decay rate for these measures on Patterson–Sullivan manifolds.

Remark: The above suggests an equivalent definition of a Patterson–Sullivan manifold as a negatively curved manifold L with a choice of uniform tiling on its universal cover such that the induced Patterson–Sullivan measures have full support and no atoms.

2.1. Examples: It is easy to see from the definition that any manifold which is locally a rank one symmetric space off of a compact set is a Patterson–Sullivan manifold.

We will now construct some examples of foliations $(N, \mathcal{F}_N)$ with transverse quasi-invariant measures ν such that ν-almost every leaf is a Patterson–Sullivan manifold.

Assume for the moment that for almost every leaf $L \in \mathcal{F}_N$ and any $x \in \tilde{L}$, there exists constants $C_3, C_4 > 1$ such that

$$(2.5) \qquad C_3 e^{-\delta d(x,p)} \le \liminf_{R\to\infty} \frac{\operatorname{Vol} S(x,R)}{\operatorname{Vol} S(p,R)} \le \limsup_{R\to\infty} \frac{\operatorname{Vol} S(x,R)}{\operatorname{Vol} S(p,R)} \le C_4 e^{\delta d(x,p)},$$

where $p \in \tilde{L}$ is the arbitrarily chosen basepoint and $\delta < h(g_L)$. From 2.5, it follows that there exists R (depending on x) such that

$$(2.6) \qquad \operatorname{Vol}(S(p,r)) \frac{1}{2} C_3 e^{-\delta d(x,p)} \le \operatorname{Vol}(S(x,r)) \le 2C_4 \operatorname{Vol}(S(p,r)) e^{\delta d(x,p)},$$

for all $r > R$.

We will show that for every $x \in \tilde{L}$ there are constants C_5, C_6 such that

$$C_5 e^{-\delta d(x,p)} \le c(x) \le C_6 e^{\delta d(x,p)},$$

which implies that L is a Patterson–Sullivan manifold.

Notice that from the definition of $c(x)$ and the choice of the sequence s_i,

$$c(x) = \lim_{s_i \to h(g_L)^+} \nu_x^{s_i}(\partial \tilde{L}) = \lim_{s_i \to h(g_L)^+} \frac{g_{s_i}(x)}{g_{s_i}(p)}.$$

Therefore

$$\begin{aligned}
c(x) &= \lim_{s_i \to h(g_L)^+} \frac{g_{s_i}(x)}{g_{s_i}(p)} \\
&\le \lim_{s_i \to h(g_L)^+} \frac{(C_1 e^{s_i C_2})^2 \int_0^\infty q(t) e^{-s_i t} \operatorname{Vol}(S(x,t))dt}{\int_0^\infty q(t) e^{-s_i t} \operatorname{Vol}(S(p,t))dt} \\
&= (C_1 e^{h(g_L)C_2})^2 \lim_{s_i \to h(g_L)^+} \frac{\int_0^\infty q(t) e^{-s_i t} \operatorname{Vol}(S(x,t))dt}{\int_0^\infty q(t) e^{-s_i t} \operatorname{Vol}(S(p,t))dt} \\
&= (C_1 e^{h(g_L)C_2})^2 \lim_{s_i \to h(g_L)^+} \frac{\int_R^\infty q(t) e^{-s_i t} \operatorname{Vol}(S(x,t))dt}{\int_R^\infty q(t) e^{-s_i t} \operatorname{Vol}(S(p,t))dt} \\
&\le (C_1 e^{h(g_L)C_2})^2 \lim_{s_i \to h(g_L)^+} \frac{\int_R^\infty q(t) e^{-s_i t} 2C_4 \operatorname{Vol}(S(p,t)) e^{\delta d(x,p)} dt}{\int_R^\infty q(t) e^{-s_i t} \operatorname{Vol}(S(p,t))dt} \qquad \text{from (2.6)} \\
&= (C_1 e^{h(g_L)C_2})^2 2C_4 e^{\delta d(x,p)} = C_6 e^{\delta d(x,p)},
\end{aligned}$$

where the fourth line holds because the integrals from 0 to R are bounded as s approaches $h(g_L)$ while the integrals from R to ∞ diverge. The inequality $C_5 e^{-\delta d(x,p)} \leq c(x)$ follows in the same manner, finishing the claim that L is a Patterson–Sullivan manifold.

Therefore it is sufficient to find examples where condition (2.5) is satisfied.

Suppose for some quasi-invariant measure ν almost every leaf L has a group G_L acting on $\tilde{L}$ by isometries with respect to the metric g_L. If G_L has a compact fundamental domain on $\tilde{L}$, then we claim condition (2.5) is satisfied. To see this we observe that the ratio

$$f(x,y,R) = \frac{\operatorname{Vol} S(y,R)}{\operatorname{Vol} S(x,R)}$$

is continuous in x and y and bounded from above and below independently of R. Since it is invariant under the action of G_L on the first two coordinates, $f(x,y,R)$ is bounded independently of x and y. The claim follows.

Examples of such foliations with cocompact group actions include ones where ν almost every leaf is a compact of negative curvature. Also, any foliation with ν almost every leaf a homogeneous space of negative curvature satisfy the Patterson–Sullivan condition. Given a product of rank one symmetric spaces $X = X_1 \times \cdots \times X_N$ where X_i is one of $\mathbb{H}^{n_i}, \mathbb{CH}^{n_i}, \mathbb{QH}^{n_i}$, or $\mathrm{Ca}\,\mathbb{H}^2$, then for certain combinations of factors there always exist irreducible cocompact lattices in $\operatorname{Iso}(X)$ (see Theorem 9.2.6 in [Eb96]). In that case, $\Gamma\backslash X$ is nontrivially foliated by the negatively curved leaves corresponding to the factors X_i. More generally, for any Lie group G of noncompact type and a closed subgroup H and for any uniform lattice Γ of G, if a closed subgroup $Z \subset G$ is such that $Z/(H \cap Z)$ has negative curvature in the metric induced from a left invariant metric on G/H, then locally homogeneous space $\Gamma\backslash G/H$ is foliated by the left cosets of $Z/(H \cap Z)$. By the above, these leaves will be Patterson–Sullivan manifolds. In the case of a noncompact semisimple group G and a maximal compact subgroup H, the image in $\Gamma\backslash G/H$ of any rank one simple Lie subgroup generates such a foliation by isometrically embedded negatively curved leaves. Several totally geodesic examples of such subalgebras are described in Section 2.20 of [Eb96]. These arise as Lie triple systems. There are many other rank one subalgebras which are only isometrically embedded, but nevertheless give rise to foliations of $\Gamma\backslash G/H$ by Patterson–Sullivan manifolds.

It is clear that compact perturbations of the previous examples of foliations preserve the Patterson–Sullivan property provided that the perturbation is restricted to a set where the leaves in the support of ν do not recur infinitely often;

in other words, when the perturbed geometry on each affected leaf remains locally symmetric off a compact set. More general perturbations of the foliations will not be of this type of course, since they will affect the asymptotic geometry of large spheres, and it remains an open question whether they satisfy the Patterson–Sullivan condition.

3. The natural map F on Patterson–Sullivan manifolds

In this section we define the natural map and prove Theorem 2. Recall from the introduction that we are assuming that (L, g_L) is a Patterson–Sullivan manifold and $f\colon L \to X$ is a homeomorphism from L to a negatively curved manifold (X, g_o) whose lifts to the universal cover $\tilde{f}\colon \tilde{L} \to \tilde{X}$ are quasi-isometries. (In §4 we show that this is true in our foliation situation.) Given a lift $\tilde{f}$, it extends to a homeomorphism $\bar{f}$ between the boundaries $\partial\tilde{L}$ and $\partial\tilde{X}$. Recall that we chose a basepoint $p \in \tilde{L}$ and defined Patterson–Sullivan measures μ_x in terms of the basepoint. By pushing forward the μ_x on $\partial\tilde{L}$ we obtain new measures $\bar{f}_*\mu_x$ on $\tilde{X}$. Let $B(y,\theta) = B(p,y,\theta)$ be the Busemann function of $y \in \tilde{L}$ at $\theta \in \partial\tilde{L}$ with respect to the basepoint p on $(\tilde{L}, g_{\tilde{L}})$, and similarly let $B_o(y,\theta) = B_o(\tilde{f}(p), y, \theta)$ be the Busemann function on $(\tilde{X}, g_o)$ with respect to the basepoint $\tilde{f}(p)$ (which by abuse of notation we will also denote by p). For $x \in \tilde{L}, y \in \tilde{X}$ define the function

$$\mathcal{B}(x,y) \stackrel{\text{def}}{=} \int_{\partial\tilde{X}} B_o(y,\theta)\, d\bar{f}_*\mu_x(\theta) = \int_{\partial\tilde{L}} B_o(y, \bar{f}(\theta))\, d\mu_x(\theta).$$

Using the convexity of the Busemann function, one can show ([BCG96], Theorem 3.1) that for fixed x, the function $y \mapsto \mathcal{B}(x,y)$ has a unique critical point in $\tilde{X}$ which is its minimum.

We can now define on the universal covers a map $\tilde{F}\colon \tilde{L} \to \tilde{X}$ by

$$\tilde{F}(x) \stackrel{\text{def}}{=} \text{the unique critical point of } \mathcal{B}(x,\cdot).$$

Since for any two points $p_1, p_2 \in \tilde{X}$, $B_o(p_1, y, \theta) = B_o(p_2, y, \theta) + B_o(p_1, p_2, \theta)$, we see that $\mathcal{B}(x,\cdot)$ only changes by an additive constant when we change the basepoint of B_o. Also, $\mathcal{B}(x,\cdot)$ only changes by a multiplicative constant when we change the basepoint in the definition of μ_x. Since neither change affects the critical point of $\mathcal{B}(x,\cdot)$, $\tilde{F}$ is independent of choice of basepoints. If Γ_L and Γ_X are the discrete groups of deck transformations of the universal covers $\tilde{L} \to L$ and $\tilde{X} \to X$ respectively, then $x \mapsto \mu_x$ and B_o are Γ_L-equivariant and Γ_X-equivariant respectively, and $\tilde{f}(\gamma x) = \rho(\gamma)\tilde{f}(x)$, which implies that $\tilde{F}(\gamma x) = \rho(\gamma)\tilde{F}(x)$, where

$\rho\colon \Gamma_L \to \Gamma_X$ is the isomorphism between the fundamental groups induced by the homeomorphism f. Hence $\tilde{F}$ descends to the **natural map** $F\colon L \to X$ which is known to be C^1 (see [BCG96]).

The proof of Theorem 2 relies on the following two key lemmas.

LEMMA 3.1: *The map $\tilde{F}$ is proper.*

Proof: If not, then there would exist a sequence of points x_n tending to $\eta \in \partial\tilde{L}$ such that $\tilde{F}(x_n)$ tends to a point $z \in \tilde{X}$. Explicitly, the $\tilde{F}(x_n)$ satisfy

$$\begin{aligned}\min_{y\in\tilde{X}} \mathcal{B}(x_n, y) &= \min_{y\in\tilde{X}} \int_{\partial\tilde{L}} B_o(y, \bar{f}(\theta))d\mu_{x_n}(\theta)\\ &= \int_{\partial\tilde{L}} B_o(\tilde{F}(x_n), \bar{f}(\theta))d\mu_{x_n}(\theta) = \mathcal{B}(x_n, \tilde{F}(x_n)).\end{aligned}$$

Our approach is to construct a sequence of points $y_n \in \tilde{X}$ such that $\mathcal{B}(x_n, y_n) < \mathcal{B}(x_n, \tilde{F}(x_n))$, contradicting the definition of $\tilde{F}$. Let $0 < \delta < 1$ be a constant such that

$$\limsup_{x\in\tilde{L}} \left| \frac{\log c(x)}{d(p,x)} \right| < \delta h(g_L).$$

Such a δ exists because L is a Patterson–Sullivan manifold. Consider the complementary sets

$$A_n^{\leq\delta} = \left\{ \theta \in \partial\tilde{L} \,|B(x_n, \theta) \leq \delta d(p, x_n) \right\},$$

and

$$A_n^{>\delta} = \left\{ \theta \in \partial\tilde{L} \,|B(x_n, \theta) > \delta d(p, x_n) \right\}.$$

First we show that $\lim_{n\to\infty} A_n^{\leq\delta} = \{\eta\}$, i.e., any sequence of points $z_n \in A_n^{\leq\delta}$ converges to η. Fix any horosphere H containing p in $\tilde{L}$ tangent to $\tau \neq \eta \in \partial\tilde{L}$ and let h_n be the unique point on H closest to x_n. Then since x_n converges to η, it follows that h_n converges to a point $h \in H$. Notice that $B(\tau, x_n) = d(H, x_n) \sim d(h, x_n)$ for large n. By the triangle inequality, $d(p, x_n) \leq d(h, x_n) + d(p, h)$. Hence there is an N such that $B(\tau, x_n) > \delta d(p, x_n)$ for $n > N$. This implies that for all $n > N$, $\tau \notin A_n^{\leq\delta}$, which completes the claim.

Let η_n be the endpoint of the geodesic ray starting at the origin $p \in \tilde{L}$ and passing through x_n, and set $\tau_n = \bar{f}(\eta_n)$. Notice that $\tau_n \in \bar{f}\left(A_n^{\leq\delta}\right)$ since $B(x_n, \eta_n) = -d(p, x_n)$. For any $\theta \in \partial\tilde{X}$, let γ_θ be the unique geodesic ray between the origin $p \in \tilde{X}$ and θ. Set

$$t_n = \sup\{t\colon d_o(\gamma_\theta(t), \gamma_{\tau_n}(t)) \leq 1 \;\; \forall\theta \in \bar{f}\left(A_n^{\leq\delta}\right)\}.$$

Since the sets $A_n^{\leq\delta}$ shrink down to η as $n \to \infty$ and $\bar{f}$ is a homeomorphism, the sets $\bar{f}\left(A_n^{\leq\delta}\right)$ shrink down to $\bar{f}(\eta)$. From this one sees easily that $t_n \to \infty$ as $n \to \infty$. Let $y_n = \gamma_{\tau_n}(t_n)$, and notice that $d_o(p, y_n) = t_n$, so $d_o(p, y_n) \to \infty$ as $n \to \infty$, a fact we will use later. Choose $\theta_n \in \bar{f}\left(A_n^{\leq\delta}\right)$ such that $B_o(y_n, \theta_n) = \max_{\theta \in \bar{f}\left(A_n^{\leq\delta}\right)} B_o(y_n, \theta)$.

Since the horosphere $H_n = (B_o)^{-1}(\cdot, \theta_n)(0)$ is a C^2 limit of geodesic spheres, the sphere $S(\gamma_{\theta_n}(t_n), t_n)$ about $\gamma_{\theta_n}(t_n)$ of radius t_n is contained in the interior of the horoball with boundary H_n. Also, y_n is in the interior of the closed ball about $\gamma_{\theta_n}(t_n)$ of radius t_n. By the triangle inequality,

$$d_o(y_n, H_n) \geq d_o(y_n, S(\gamma_{\theta_n}(t_n), t_n)) \geq t_n - d_o(\gamma_{\theta_n}(t_n), y_n) \geq t_n - 1.$$

By definition of the Busemann function, $B_o(y_n, \theta_n) = -d_o(y_n, H_n) \leq 1 - t_n$, and by the choice of θ_n we can estimate

$$\int_{A_n^{\leq\delta}} B_o(y_n, \bar{f}(\theta)) d\mu_{x_n} \leq (1 - t_n) \int_{A_n^{\leq\delta}} d\mu_{x_n} = (1 - t_n)\mu_{x_n}(A_n^{\leq\delta}).$$

Also, $B_o(y_n, \bar{f}(\theta)) \leq d_o(p, y_n) = t_n$ for any θ, so

$$\int_{A_n^{>\delta}} B_o(y_n, \bar{f}(\theta)) d\mu_{x_n} \leq t_n \mu_{x_n}(A_n^{>\delta}).$$

Since $\mu_{x_n}(A_n^{\leq\delta}) = \left(1 - \mu_{x_n}(A_n^{>\delta})\right)$, summing gives

$$\begin{aligned}\mathcal{B}(x_n, y_n) &= \int_{A_n^{\leq\delta}} B_o(y_n, \bar{f}(\theta)) d\mu_{x_n} + \int_{A_n^{>\delta}} B_o(y_n, \bar{f}(\theta)) d\mu_{x_n} \\ &\leq 1 - t_n + (2t_n - 1)\mu_{x_n}(A_n^{>\delta}).\end{aligned}$$

Lastly, we show that $\mu_{x_n}(A_n^{>\delta}) \to 0$ as $n \to \infty$. By (2.4) and since $c(p) = 1$,

$$\begin{aligned}\mu_{x_n}(A_n^{>\delta}) &= \int_{B(x_n,\theta) > \delta d(p,x_n)} \frac{\exp\{-h(g_L)B(x_n, \theta)\}}{c(x_n)} d\mu_p(\theta) \\ &\leq \frac{\exp\{-\delta h(g_L) d(p, x_n)\}}{c(x_n)} \mu_p(A_n^{>\delta}) \\ &\leq \frac{\exp\{-\delta h(g_L) d(p, x_n)\}}{c(x_n)},\end{aligned}$$

and the last quantity goes to 0 as $n \to \infty$ by the choice of δ.

Since $t_n \to \infty$ as $n \to \infty$, we conclude that $\lim_{n\to\infty} \mathcal{B}(x_n, y_n) = -\infty$. However, we assumed that $\tilde{F}(x_n)$ converges to z, hence for any $\epsilon > 0$ and all sufficiently large n, the continuity of B_o and the estimate $B_o(z, \theta) \geq -d(p, z)$ imply

$$\mathcal{B}(x_n, \tilde{F}(x_n)) > \mathcal{B}(x_n, z) - \epsilon > -d_o(p, z) - \epsilon.$$

The last term is bounded, which contradicts the minimality of $\tilde{F}(x_n)$ since we have $\mathcal{B}(x_n, \tilde{F}(x_n)) < \mathcal{B}(x_n, y_n)$. ∎

LEMMA 3.2: *$\tilde{F}$ extends continuously to the homeomorphism $\bar{f}$ on the boundary.*

Proof: If not, then by Lemma 3.1 there exists a geodesic γ in $\tilde{L}$ and a sequence $x_n = \gamma(t_n)$ converging to $\xi \in \partial\tilde{L}$ such that $\tilde{F}(x_n)$ converges to $\bar{f}(\eta) \in \partial\tilde{X}$ for some $\eta \neq \xi \in \partial\tilde{L}$.

Consider

$$\begin{aligned} A^+_{o,n} &= \left\{\theta \in \partial\tilde{L} \,\middle|\, B_o(\tilde{F}(x_n), \bar{f}(\theta)) \geq 0\right\}, \\ A^{>\delta}_{o,n} &= \left\{\theta \in \partial\tilde{L} \,\middle|\, B_o(\tilde{F}(x_n), \bar{f}(\theta)) > \delta d_o\left(p, \tilde{F}(x_n)\right)\right\}, \quad \text{and} \\ A^-_{o,n} &= \left\{\theta \in \partial\tilde{L} \,\middle|\, B_o(\tilde{F}(x_n), \bar{f}(\theta)) < 0\right\}. \end{aligned}$$

We will show that $\mathcal{B}(x_n, \tilde{F}(x_n))$ is nonnegative and derive a contradiction. Recall from the definition of B_o that for all $\theta \in \partial\tilde{L}$,

$$d_o(p, \tilde{F}(x_n)) \geq B_o(\tilde{F}(x_n), \bar{f}(\theta)) \geq -d_o(p, \tilde{F}(x_n)).$$

From this we can estimate that for all n,

$$\begin{aligned} \mathcal{B}(x_n, \tilde{F}(x_n)) &\geq \int_{A^-_{o,n}} B_o(\tilde{F}(x_n), \bar{f}(\theta)) d\mu_{x_n} + \int_{A^{>\delta}_{o,n}} B_o(\tilde{F}(x_n), \bar{f}(\theta)) d\mu_{x_n} \\ &\geq -d_o\left(p, \tilde{F}(x_n)\right) \mu_{x_n}(A^-_{o,n}) + \delta d_o\left(p, \tilde{F}(x_n)\right) \mu_{x_n}(A^{>\delta}_{o,n}) \\ &= d_o\left(p, \tilde{F}(x_n)\right) \left(\delta\mu_{x_n}(A^{>\delta}_{o,n}) - \mu_{x_n}(A^-_{o,n})\right). \end{aligned} \tag{3.1}$$

As in the proof of the previous lemma, $\lim_{n\to\infty} A^{>\delta}_{o,n} = \partial\tilde{L} \setminus \{\eta\}$ and $\bigcap_n A^-_{o,n} = \{\eta\}$. Since eventually $\xi \in A^{>\delta}_{o,n}$, one can check that μx_n does not tend to the Dirac measure concentrated at η. It follows that $\lim_n \mu_{x_n}(A^{>\delta}_{o,n}) = 1$ and $\lim_n \mu_{x_n}(A^-_{o,n}) = 0$. In particular, for sufficiently large n,

$$\delta\mu_{x_n}(A^{>\delta}_{o,n}) > \mu_{x_n}(A^-_{o,n}),$$

which by inequality (3.1) implies that $\mathcal{B}(x_n, \tilde{F}(x_n)) \geq 0$. However, in Lemma 3.1 we showed the existence of a sequence y_n such that $\mathcal{B}(x_n, y_n)$ tends to $-\infty$, contradicting the minimality of $\tilde{F}$. ∎

Here we restate Theorem 2 for the convenience of the reader.

THEOREM 3.3: *Let L be a Patterson–Sullivan manifold and $f\colon L \to X$ a homeomorphism from L to a negatively curved manifold such that some lift $\tilde{f}\colon \tilde{L} \to \tilde{X}$ is a quasi-isometry. For $\tilde{F}\colon \tilde{L} \to \tilde{X}$ the natural map as defined above, $\tilde{F}$ is a proper surjection.*

Proof of Theorem 2: By the previous lemma we may treat $\tilde{F}$ as a continuous map from $\tilde{L} \cup \partial \tilde{L}$ to $\tilde{X} \cup \partial \tilde{X}$, i.e., a map from a closed topological ball to another closed ball. But any such map which has non-zero degree on the boundary is surjective. ∎

4. The natural map on the leaves of a compact foliation

We now return to our foliation setup. For the remainder of the paper, (L, g_L, d_L) is a leaf of $(N, \mathcal{F}_N)$ and (X_L, g_o, d_o) is its image under the leaf-preserving homeomorphism $f\colon (N, \mathcal{F}_N) \to (M, \mathcal{F}_M)$ which we have assumed is C^1 when restricted to leaves with transversally C^0 leafwise derivatives. Our goal is to construct the natural map $\tilde{F}\colon \tilde{L} \to \tilde{X}_L$ and apply Theorem 2 to it. For this we need to know that the lifts of f to the universal covers extend to boundary homeomorphisms at infinity; this will hold once we show that the lifts are quasi-isometries.

LEMMA 4.1: *The restriction of $f\colon (N, \mathcal{F}_N) \to (M, \mathcal{F}_M)$ to each leaf is a quasi-isometry.*

Proof: Consider any two sequences of points x_i, y_i in a fixed leaf L such that $d_L(x_i, y_i) \to 0$ in L. By compactness of N, after passing to convergent subsequences we may assume x_i and y_i both converge to a point p. We conclude from the continuity of f that $f(x_i)$ and $f(y_i)$ converge to the point $f(p)$. Since f is leaf preserving, $f(x_i)$ and $f(y_i)$ must eventually lie in the same plaque so $d_o(f(x_i), f(y_i)) \to 0$. By applying this argument to f^{-1} we conclude that $d_L(x_i, y_i) \to 0$ if and only if $d_o(f(x_i), f(y_i)) \to 0$.

Suppose for some pair of sequences $x_i, y_i \in L$ we have

$$\limsup_i \frac{d_o(f(x_i), f(y_i))}{d_L(x_i, y_i)} = \infty$$

and let α_i be a minimizing geodesic path in L between x_i and y_i. Assume that $d_L(x_i, y_i)$ always exceeds a fixed constant ϵ. By considering points of maximum dilation, for any numbers $c_i \leq d_L(x_i, y_i)$ there exist points $p_i, q_i \in \alpha_i$ with

$d_L(p_i, q_i) = c_i$ and

$$d_o(f(p_i), f(q_i)) \geq c_i d_o(f(x_i), f(y_i)) \geq \epsilon c_i \frac{d_o(f(x_i), f(y_i))}{d_L(x_i, y_i)}.$$

Since we assumed

$$\limsup_i \frac{d_o(f(x_i), f(y_i))}{d_L(x_i, y_i)} = \infty,$$

choosing

$$c_i = \min\left\{ d_L(x_i, y_i), \sqrt{\frac{d_L(x_i, y_i)}{d_o(f(x_i), f(y_i))}} \right\}$$

implies that there is a subsequence of the p_i, q_i such that $d_L(p_i, q_i) \to 0$ and $d_o(f(p_i), f(q_i)) \geq \epsilon$. This contradicts our earlier result, so

$$\limsup_i \frac{d_o(f(x_i), f(y_i))}{d_L(x_i, y_i)}$$

must stay bounded when $d_L(x_i, y_i) > \epsilon$. As a consequence,

$$K^{-1} d_o(f(x), f(y)) \leq d_L(x, y)$$

for $d_L(x, y) \geq \epsilon$. By considering what happens to the complement of $B_{d_L}(x, \epsilon)$ for fixed x, it follows that $f(B_{d_L}(x, \epsilon)) \subset B_{d_o}(f(x), K\epsilon)$; i.e., whenever $d_L(x, y) < \epsilon$ then $d_o(f(x), f(y)) < K\epsilon$. Hence for all $x, y \in L$,

$$-\epsilon + K^{-1} d_o(f(x), f(y)) \leq d_L(x, y).$$

The case when

$$\limsup_i \frac{d_L(x_i, y_i)}{d_o(f(x_i), f(y_i))} = \infty$$

can be treated analogously by using f^{-1} to reverse the situation. This yields $d_L(x, y) \leq K d_o(f(x), f(y))$ when $d_o(f(x), f(y)) > \epsilon$. Again showing that for $x, y \in L$,

$$d_L(x, y) \leq K d_o(f(x), f(y)) + \epsilon. \quad \blacksquare$$

PROPOSITION 4.2: *Any lift $\tilde{f}$: $\tilde{L} \to \tilde{X}_L$ of the restriction of f to a leaf is a surjective quasi-isometry between the universal covers.*

Proof: We will use a set of sufficient conditions given by Y. Minsky. Since f: $L \to X$ is a continuous map between complete, locally compact, connected path-metric spaces, by Lemma 4.4 of [Min94] we need only verify the following four criteria:

Q1. The map f is a proper, surjective, homotopy equivalence.

Q2. The map f is a (K,ϵ) quasi-isometry, for some $K \geq 1$ and $\epsilon \geq 0$.

Q3. Any lift $\tilde{f}$: $\tilde{L} \to \tilde{X}_L$ is Lipschitz in the large.

Q4. For every $B > 0$ there exists an $A > 0$ such that, if $x \in L$ and $\beta \subset X_L$ is a loop through $f(x)$ of length $l_{X_L}(\beta) < B$, then there is a loop $\alpha \subset L$ through x with $l_L(\alpha) < A$, and $f(\alpha)$ is homotopic to β.

Condition Q1 holds since the restriction of f to L is a homeomorphism. Condition Q2 is the statement of Lemma 4.1. Since we assumed f is leafwise C^1 and the leaf metrics are transversally continuous, the compactness of N implies that the derivatives of f, and hence $\tilde{f}$, are bounded, yielding Condition Q3. It remains to verify Condition Q4.

Assume by way of contradiction that there is a sequence of loops β_i in the leaf X_L with length less than some fixed B such that all loops α_i in L with $f(\alpha_i)$ in the same homotopy class as the β_i have $l_L(\alpha_i) \to \infty$. In particular, we may assume that α_i is a piecewise smooth curve through x_i which is the shortest closed curve in its homotopy class, and by choosing a subsequence that $l_i = \text{length}(\alpha_i) > i \to \infty$. Note that the injectivity radius of X_L is bounded below by some constant $1 > C > 0$ since plaque sizes are bounded on M. Chop β_i into i pieces $\{P_j\}_{j=1}^{i}$, each of length

$$\epsilon_i = \frac{\text{length}(\beta_i)}{i},$$

which goes to zero as $i \to \infty$ since $\text{length}(\beta_i) < B$. We claim that for some j, $\text{diam}(f^{-1}P_j) \geq C$. For if not, let a_j, a_{j+1} be the endpoints of P_j, and notice that $d(f^{-1}a_j, f^{-1}a_{j+1}) \leq \text{diam}(f^{-1}P_j) < C$ for each j. Let c_j be the unique minimizing geodesic arc between $f^{-1}(a_j)$ and $f^{-1}(a_{j+1})$, which we note is homotopic to $f^{-1}P_j$ for i large enough. Then $\text{length}(c_j) \leq d(f^{-1}a_j, f^{-1}a_{j+1}) < C$ implies that

$$\sum_{j=1}^{i} \text{length}(c_j) < Ci < Cl_i < l_i,$$

and so the broken geodesic $c_1 \cup c_2 \cup \cdots \cup c_j$ is a curve through x_i which is homotopic to α_i, but strictly shorter than it, a contradiction.

We conclude that for each i there is a number $j(i) \leq i$ and a piece $P_{j(i)}$ of β_i with $\text{diam}(f^{-1}P_j) \geq C$. Taking limits in the compact manifold M, there is a subsequence of the β_i such that $\lim_{i\to\infty} P_{j(i)}$ is a single point, while the same subsequence of $f^{-1}\left(P_{j(i)}\right)$ has diameter bounded away from zero. However, this contradicts the fact that f is a homeomorphism, verifying Condition Q4 and the proposition. ∎

Theorem 2 now tells us that the natural maps constructed leafwise on $(N, \mathcal{F}_N)$ are surjective. We combine these leafwise natural maps into a global map F: $N \to M$. Since f was assumed to be a bijection between the leaf spaces of $\mathcal{F}_M$ and $\mathcal{F}_N$, F is as well. Because F is defined in terms of leafwise Busemann functions, the fact that horospheres are the continuous limit of geodesic spheres in each leaf implies by transverse continuity of the metrics that F is the limit of continuous functions on M, and hence is measurable. We summarize with

COROLLARY 4.3: *The map F: $(N, \mathcal{F}_N) \to (M, \mathcal{F}_M)$ is a measurable foliation-preserving surjection which is C^1 when restricted to leaves.*

5. Proof of the Main Theorem

Recall that the foliation $(N, \mathcal{F}_N)$ possesses a holonomy quasi-invariant measure ν and $d\mu_N$ is the globally defined and finite measure given locally by $d\nu \times dg_L$. The push-forward measure $F_*\nu$ is the measure ν_o referred to in the statement of the Main Theorem. It is holonomy quasi-invariant, and so on M we have the globally defined and finite measure $d\mu_M$ given locally by $d\mu_M = dF_*\nu \times dg_o$. When ν is actually holonomy invariant, then in fact $\nu_o = F_*\nu = f_*\nu$ (since f and F are homotopic), and so in this case the measure ν_o can be described without reference to the natural map F.

Our first ingredient is a foliated version of the coarea formula from geometric measure theory.

PROPOSITION 5.1 (Foliated coarea formula): *Let $\operatorname{Jac} F$ be the leafwise Jacobian of F and $p(F, y) = \#\{F^{-1}(y)\}$ the leaf-wise preimage counting function (possibly infinite). Then*

$$\int_N |\operatorname{Jac} F(x)| d\mu_N(x) = \int_M p(F, y) d\mu_M(y).$$

Proof: Recall that $\{U_i\}_{i=1}^m$ is a covering of N by flow boxes with local cross sections T_i. We first indicate why we may assume without loss of generality that F restricted to each such T_i is injective. Let $\{t_j\}$ be an infinite set of distinct points in $T_i \cap L$. Since the plaques have leaf-wise inscribed diameter bounded from below away from zero and the points t_j lie in different plaques, the points t_j are unbounded in the metric on the leaf L. From Lemma 3.2, F cannot be constant on such a set of points t_j. It follows that if $F_{\rceil T_i}$ is not injective, then we may assume that T_i is a finite union of open sets on which F is injective. These open sets naturally introduce a subcovering of the flow box U_i by sets with a product structure. Taking all such sets over all $i \leq m$, we get a finite refinement of our covering of N by flowboxes (which we continue to write as $\{U_i\}_{i=1}^m$) for which the restriction of F to the local cross sections T_i is injective.

Now we let $\{U_i\}_{i=1}^m$ (resp. $\{O_j\}_{j=1}^l$) be a covering of N (resp. M) by flow boxes, T_i (resp. S_j) a local cross section (i.e., a transversal contained in U_i (resp. O_j) with one point on each plaque), and $\{\Psi_i\}_{i=1}^m$ (resp. $\{\Phi_j\}_{j=1}^l$) a partition of unity on the atlas of flow boxes $\{U_i\}_{i=1}^m$ (resp. $\{O_j\}_{j=1}^l$). Also, when $t \in T_i$, we denote by L_t the plaque passing through t, and for $s \in S_j$, L_s^j denotes the plaque of O_j through s. We prove the coarea formula first on a single flow box U_i in $(N, \mathcal{F}_N)$. By first applying the usual coarea formula to the plaques, and then using change of variables, we get that

$$\begin{aligned}\int_{T_i}\int_{L_t} &\Psi_i(x)|\operatorname{Jac} F(x)|dg_L(x)d\nu(t)\\ &= \int_{T_i}\int_{F(L_t)} \sum_{x\in\left\{F^{-1}_{|L_t}(y)\right\}} \Psi_i(x)dg_o(y)d\nu(t)\\ &= \int_{F(T_i)}\int_{F(L_{F^{-1}(s)})} \sum_{x\in\left\{F^{-1}_{|L_{F^{-1}(s)}}(y)\right\}} \Psi_i(x)dg_o(y)dF_*\nu(s).\end{aligned}$$

Now we break up the inner integral over all flow boxes $\{O_j\}_{j=1}^l$ in M, and rewrite the previous line as

$$= \sum_{j=1}^l \int_{F(T_i)}\int_{F(L_{F^{-1}(s)})\cap O_j} \Phi_j(y) \sum_{x\in T_s(y)} \Psi_i(x)dg_o(y)dF_*\nu(s).$$

where

$$T_s(y) = \left\{F^{-1}_{|L_{F^{-1}(s)}}(y)\right\}.$$

Since F is proper on each leaf, $F(T_i)$ is finite in each plaque of O_j. Hence one can show (via a Borel selection process) that there are measurable sets W_j^k with at most one point in each plaque of O_j such that $F(T_i)\cap O_j = \bigcup_{k\geq 1} W_j^k$ (disjoint union). The last line then becomes

$$\begin{aligned}&= \sum_{j=1}^l \sum_{k\geq 1} \int_{W_j^k}\int_{F(L_{F^{-1}(s)})\cap O_j} \Phi_j(y) \sum_{x\in T_s(y)} \Psi_i(x)dg_o(y)dF_*\nu(s).\\ &= \sum_{j=1}^l \int_{F(U_i)\cap O_j} \Phi_j(y) \sum_{x\in\left\{F^{-1}_{|U_i}(y)\right\}} \Psi_i(x)d\mu_M(y).\end{aligned}$$

$$= \int_{F(U_i)} \sum_{x \in \left\{F^{-1}_{|U_i}(y)\right\}} \Psi_i(x) d\mu_M(y).$$

Summing over all flow boxes U_i gives

$$\begin{aligned}\int_N |\operatorname{Jac} F(x)| d\mu_N(x) &= \sum_{i=1}^m \int_{T_i} \int_{L_t} \Psi_i(x) |\operatorname{Jac} F(x)| dg_L(x) d\nu(t) \\ &= \sum_{i=1}^m \int_{F(U_i)} \sum_{x \in \left\{F^{-1}_{|U_i}(y)\right\}} \Psi_i(x) d\mu_M(y) \\ &= \int_M \sum_{\{U_i | y \in F(U_i)\}} \sum_{x \in \left\{F^{-1}_{|U_i}(y)\right\}} \Psi_i(x) d\mu_M(y) \\ &= \int_M p(F, y) d\mu_M(y). \qquad \blacksquare\end{aligned}$$

We will also need the following important proposition from [BCG96] which gives an estimate on the Jacobian of the natural map; it applies in our case since the proof does not rely on the compactness of L.

PROPOSITION 5.2 (Proposition 5.2 of [BCG96]): *Fix a leaf L. Recall that $h(g_L)$ and $h(g_o)$ are the volume growth entropies of L and $f(L)$ with respect to the metrics g_L and g_o. Then*

1. $|\operatorname{Jac} F(x)| \leq \left(\frac{h(g_L)}{h(g_o)}\right)^n$ *for every $x \in L$, and*
2. *if for some $x \in L$, $|\operatorname{Jac} F(x)| = \left(\frac{h(g_L)}{h(g_o)}\right)^n$, then the differential dF_x of F at x is a homothety of ratio $\frac{h(g_L)}{h(g_o)}$.*

Proof of the Main Theorem: Let $(N_\alpha, \mathcal{F}_{N_\alpha}, \nu_\alpha)_{\alpha \in A}$ be the ergodic decomposition of $(N, \mathcal{F}_N, \nu)$ and $(M_\alpha, \mathcal{F}_{M_\alpha}, f_*\nu_\alpha)_{\alpha \in A}$ the corresponding ergodic decomposition of $(M, \mathcal{F}_M, f_*\nu)$. (We use here that the foliation defines on a cross section T a countable equivalence relation in the sense of [FM75]. This equivalence relation decomposes (up to ν measure zero) into a continuous sum of ergodic equivalence relations, and ν is a continuous sum $\nu = \int_A \nu_\alpha d\alpha$ (see [FM75], §3). By transverse quasi-invariance, this induces a decomposition of $(N, \mathcal{F}_N, \nu)$ into ergodic components.)

By Corollary 4.3 the natural map F is leafwise surjective. Therefore, applying Propositions 5.1 and 5.2 to the foliations of the ergodic components, we obtain the inequalities

$$\int_{M_\alpha} d\mu_{M_\alpha} \le \int_{M_\alpha} p(F,y)d\mu_{M_\alpha} = \int_{N_\alpha} |\operatorname{Jac} F(x)| d\mu_{N_\alpha}$$
$$\le \int_{N_\alpha} \left(\frac{h_{\rfloor N_\alpha}(g_L)}{h_{\rfloor M_\alpha}(g_o)} \right)^n d\mu_{N_\alpha},$$

where $d\mu_{N_\alpha}$ is locally $dg_L \times d\nu_\alpha$ and similarly for $d\mu_{M_\alpha}$. Thus

$$\int_{M_\alpha} h_{\rfloor M_\alpha}(g_o)^n d\mu_{M_\alpha} \le \int_{N_\alpha} h_{\rfloor N_\alpha}(g_L)^n d\mu_{N_\alpha}$$

since $h_{\rfloor M_\alpha}(g_o)^n$ is constant on M_α. Integrating with respect to α gives the desired result that

$$\int_M h(g_O)^n d\mu_M \le \int_N h(g_L)^n d\mu_N. \tag{5.1}$$

In the case that equality holds in (5.1), we actually have that

$$h_{\rfloor M_\alpha}(g_o)^n \int_{M_\alpha} d\mu_{M_\alpha} = h_{\rfloor N_\alpha}(g_L)^n \int_{N_\alpha} d\mu_{N_\alpha}$$

for almost every $\alpha \in A$.

Since $|\operatorname{Jac} F|_{N_\alpha} \le \left(\frac{h_{\rfloor N_\alpha}(g_L)}{h_{\rfloor M_\alpha}(g_o)} \right)^n$ and

$$\int_{M_\alpha} d\mu_{M_\alpha} \le \int_{N_\alpha} |\operatorname{Jac} F| d\mu_{N_\alpha} \le \left(\frac{h_{\rfloor N_\alpha}(g_L)}{h_{\rfloor M_\alpha}(g_o)} \right)^n \int_{N_\alpha} d\mu_{N_\alpha} = \int_{M_\alpha} d\mu_{M_\alpha},$$

we see that $|\operatorname{Jac} F|_{N_\alpha} = \left(\frac{h_{\rfloor N_\alpha}(g_L)}{h_{\rfloor M_\alpha}(g_o)} \right)^n$ μ_{N_α}-almost everywhere and hence by Fubini–Tonelli, $|\operatorname{Jac} F|_{N_\alpha} = \left(\frac{h_{\rfloor N_\alpha}(g_L)}{h_{\rfloor M_\alpha}(g_o)} \right)^n$, dg_L-almost everywhere, on ν_α-almost every leaf. Since $|\operatorname{Jac} F|_{N_\alpha}$ is continuous on each leaf, it must be $\left(\frac{h_{\rfloor N_\alpha}(g_L)}{h_{\rfloor M_\alpha}(g_o)} \right)^n$ on ν_α-almost every leaf. We conclude by Proposition 5.2 that for ν_α-almost every leaf in N_α, $dF_{\rfloor N_\alpha}$ is a homothety of ratio $\left(\frac{h_{\rfloor N_\alpha}(g_L)}{h_{\rfloor M_\alpha}(g_o)} \right)^n$. Since ν is a continuous sum of the ν_α, this implies that on μ_N-almost every leaf, dF is a homothety of ratio $\left(\frac{h(g_L)}{h(g_o)} \right)$, concluding the proof. ∎

6. Applications

From the Main Theorem we obtain the following corollaries, in parallel with some of the applications found in [BCG95].

COROLLARY 6.1 (Foliated Mostow Rigidity): *Let $(N, \mathcal{F}_N)$ and $(M, \mathcal{F}_M)$ be two continuous foliations of compact spaces such that $\mathcal{F}_N$ possesses a finite transverse invariant measure ν. Let $f\colon M \to N$ be a foliation preserving leafwise C^1 homeomorphism and assume that almost all leaves L (resp. $f(L)$) in the support of ν (resp. $f_*\nu$) carry metrics locally isometric to a fixed n-dimensional symmetric space $(\tilde{X}_0, g_o)$ (resp. $(\tilde{X}_1, g_1)$) of negative curvature and dimension greater than 2. Then ν-almost every leaf (L, g_o) is homothetic to $(f(L), g_1)$.*

Proof: Note that because the leaves are symmetric spaces, they are Patterson–Sullivan manifolds, so we can apply the Main Theorem. By switching the roles of $(N, \mathcal{F}_N, \nu)$ and $(M, \mathcal{F}_M, f_*\nu)$, from the Main Theorem we obtain the two inequalitites

$$\int_M h(g_o)^n d\mu_M \leq \int_N h(g_1)^n d\mu_N \quad \text{and} \quad \int_N h(g_1)^n d\mu_N \leq \int_M h(g_o)^n d\mu_M.$$

Thus we are in the case of equality, and so the desired conclusion follows from the Main Theorem. ∎

Remarks:

1. This yields the usual Mostow Rigidity Theorem when both foliations consist of a single compact leaf.
2. Pansu and Zimmer [PZ89] have also obtained a foliated version of Mostow's rigidity theorem, although their assumptions, conclusion, and method of proof all differ from ours.

Now, letting Ric denote the Ricci scalar curvature (i.e., the trace of the curvature tensor) we obtain

COROLLARY 6.2: *Assume the hypotheses of Theorem 1, except that the leaves of $(M, \mathcal{F}_M)$ are all locally isometric to real hyperbolic space of constant curvature -1. Then*

$$\mathrm{Ric}(g_L) \geq -(n-1)g_L \Longrightarrow \int_M d\mu_M \geq \int_N d\mu_N.$$

Moreover, if $(M, \mathcal{F}_M)$ is ergodic, then equality holds if and only if almost every leaf of $(N, \mathcal{F}_N)$ is locally isometric to real hyperbolic space of constant curvature -1.

Proof: Bishop's comparison theorem gives us that

$$\mathrm{Ric}(g_L) \geq -(n-1)g_L \Longrightarrow h(g_L) \leq (n-1) = h(g_o).$$

The conclusions now follow easily from the Main Theorem. Note that we must assume ergodicity for the equality case so that

$$\int_M d\mu_M = \int_N d\mu_N \Rightarrow \int_M h(g_o)^n d\mu_M = \int_N h(g_L)^n d\mu_N. \qquad \blacksquare$$

Gromov has defined an important invariant, the minimal volume of a space Y. More precisely, if $K(g)$ denotes the sectional curvature of the metric g, then $\min \mathrm{Vol}(Y) = \inf\{Vol(Y,g)\colon |K(g)| \leq 1\}$. We can make a similar definition for a foliated space $(M, \mathcal{F}_M, \nu)$ with holonomy invariant measure, namely

$$\min \mathrm{Vol}(M, \mathcal{F}_M, \nu) = \inf\{\int_M d\mu_g\colon |K(g)| \leq 1\}$$

where the infimum is over all leafwise Riemannian metrics g with sectional curvatures $K(g)$. Then we get the following

COROLLARY 6.3: *With the same hypotheses as in the previous corollary,*

$$\min \mathrm{Vol}(M, \mathcal{F}_M, \nu) \geq \int_M d\mu_M.$$

Lastly, we extend a corollary obtained by Besson–Courtois–Gallot about Einstein metrics (see Theorem 9.6 of [BCG95]) to the foliated case.

COROLLARY 6.4: *If $\mathcal{F}_M$ is a continuous 4-dimensional foliation of a compact space M with a finite transverse invariant measure ν such that each leaf L is endowed with a real hyperbolic metric g_o, then (up to multiplication by a constant) that is the only family of negatively curved Einstein metrics admitted by the foliation.*

Proof: Let $(N, \mathcal{F}_N)$ be the same foliation of M but with g_L another family of Einstein metrics on the leaves L. We may split the Pfaffian $Pf(K_L)$ of the leaf-wise curvature tensor K_L into three components as

$$Pf(K_L) = |\, W_{g_L}|^2 - |\, Z_{g_L}|^2 + |\, U_{g_L}|^2$$

(see p. 161 of [Bes87]) where $|\, W_{g_L}|$ is the operator norm of the Weyl tensor W_{g_L} of the metric g_L, $|\, Z_{g_L}| = C_1 \big| \mathrm{Ricci}(g_L) - \frac{1}{n}\mathrm{scal}(g_L) g_L \big|$ for some constant C_1 where Ricci and scal are the Ricci tensor and scalar curvatures respectively, and lastly $|\, U_{g_L}| = C_2 \,\mathrm{scal}(g_L)$ for another constant C_2. Using the Connes foliated Gauss–Bonnet theorem [Con94] we have

$$\chi(\mathcal{F}_N) = \frac{1}{8\pi^2}\int_N Pf(K_L) d\mu_N = \frac{1}{8\pi^2}\int_N |\, W_{g_L}|^2 - |\, Z_{g_L}|^2 + |\, U_{g_L}|^2 d\mu_N$$

where $\chi(\mathcal{F}_N)$ is the "average Euler characteristic" of a leaf of N, a topological invariant for the foliation $\mathcal{F}_N$.

By compactness of M we may renormalize the metrics g_L such that $\mathrm{Ricci}(g_L) = -C_L g_L$ where $C_L \geq (n-1)$. Then using the above formula and the facts that $Z_g = 0$ for any Einstein metric g and $W_{g_o} = 0$ because g_o is locally conformally flat, we obtain

$$\begin{aligned}\chi(\mathcal{F}_N) &\geq \frac{1}{8\pi^2}\int_N |\, U_{g_L}|^2 d\mu_N \geq \frac{C_2^2}{8\pi^2}n^2 \int_M C_L^2 d\mu_N \\ &\geq \frac{C_2^2}{8\pi^2}n^2(n-1)^2 \int_N d\mu_N \geq \frac{C_2^2}{8\pi^2}n^2(n-1)^2 \int_M d\mu_M \\ &= \frac{1}{8\pi^2}\int_M |\, U_{g_o}|^2 d\mu_M = \chi(\mathcal{F}_M).\end{aligned}$$

We used Corollary 6.2 for the fourth inequality. Comparing both ends of the inequalities (and since $\mathcal{F}_M = \mathcal{F}_N$) shows that every inequality is an equality and hence we are in the equality case of Corollary 6.2 which implies that each g_L is homothetic to g_o. ∎

References

[BCG95] G. Besson, G. Courtois and S. Gallot, *Entropies et rigidités des espaces localement symétriques de courbure strictement négative*, Geometric and Functional Analysis **5** (1995), 731–799.

[BCG96] G. Besson, G. Courtois and S. Gallot, *Minimal entropy and mostow's rigidity theorems*, Ergodic Theory and Dynamical Systems **16** (1996), 623–649.

[Bes87] A. L. Besse, *Einstein Manifolds*, Springer-Verlag, Berlin, 1987.

[Con94] A. Connes, *Noncommutative Geometry*, Academic Press Inc., San Diego, CA, 1994.

[Eb96] P. Eberlein, *Geometry of nonpositively curved manifolds*, Chicago Lectures in Mathematics, University of Chicago Press, 1996.

[FJ93] F. T. Farrell and L. E. Jones, *Topological rigidity for compact non-positively curved manifolds*, in *Differential Geometry: Riemannian Geometry (Los Angeles, CA, 1990)*, American Mathematical Society, Providence, RI, 1993, pp. 229–274.

[FM75] J. Feldman and C. C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras. I*, Bulletin of the American Mathematical Society **81** (1975), 921–924.

[Hur94] S. Hurder, *Coarse geometry of foliations*, in *Geometric Study of Foliations (Tokyo, 1993)*, World Science Publishing, River Edge, NJ, 1994, pp. 35–96.

[Man79] A. Manning, *Topological entropy for geodesic flows*, Annals of Mathematics (2) **110** (1979), 567–573.

[Min94] Y. N. Minsky, *On rigidity, limit sets, and end invariants of hyperbolic 3-manifolds*, Journal of the American Mathematical Society **7** (1994), 539–588.

[Pat76] S. J. Patterson, *The limit set of a Fuchsian group*, Acta Mathematica **136** (1976), 241–273.

[PZ89] P. Pansu and R. Zimmer, *Rigidity of locally homogeneous metrics of negative curvature on the leaves of a foliation*, Israel Journal of Mathematics **68** (1989), 56–62.

[Sul79] D. Sullivan, *The density at infinity of a discrete group of hyperbolic motions*, Publications Mathématiques de l'Institut des Hautes Études Scientifiques **50** (1979), 225–250.

[Zim82] R. J. Zimmer, *Ergodic theory, semisimple Lie groups, and foliations by manifolds of negative curvature*, Publications Mathématiques de l'Institut des Hautes Études Scientifiques **55** (1982), 37–62.